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Exact Analysis of the Solution of Caputo Fractional Differential Equations Using Laplace Transforms

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Abstract:

The fractional differential equations (FDEs) are an extension of the classical differential equations to non-integer order, which provides powerful models of systems with memory in physics, engineering, and biology. Caputo fractional derivative has been extensively applied because it admits to the physical initial conditions. The present paper develops a strict analysis system in solving linear Caputo FDEs using the Laplace transform technique that obtains the precise solution expressed in special functions such as the Mittag-Leffler function. We describe the methodology, provide examples, and use current literature to support the methodology to be precise without having to make any numerical approximations. It is a scholar-friendly piece of work that would offer precise answers to theoretical and practical research works.

Keywords: Caputo fractional derivative, Fractional differential equations. Laplace transform, Exact analytical solutions, Mittag-Leffler function, Fractional calculus, Linear differential equations, Initial value problems, Memory effects, Anomalous diffusion.

1. Introduction

Fractional equations have enabled the investigation of the nonlocal response of multiple phenomena such as diffusion processes, electrodynamics, fluid flow, elasticity and many more [1−5]; fractional derivatives are memory operators which usually represent dissipative effects or damage. Some fundamental definitions of fractional derivatives were given by Coimbra, Davison and Essex, Riesz, Riemann-Liouville, Hadamard, Weyl, Jumarie, Grünwald-Letnikov, and Liouville-Caputo [6−8], and the properties of these derivatives are reviewed in [9]. The use of Caputo and Caputo-Fabrizio fractional derivatives is gaining importance in physics because of their specific properties, in both definitions, for a constant the derivative is zero and the initial conditions used in the fractional differential equations having a direct physical interpretation [10, 11]; however, the Liouville-Caputo fractional operator presents a singularity in its kernel. With the purpose to describe in a better way the memory effect, Caputo and Fabrizio presented a novel definition with an exponential kernel named the Caputo-Fabrizio fractional operator [10], this novel fractional operator is considered as a fractional filter. Applications of this fractional operator are given in [12−15].

The constructions of the exact and explicit solutions of the partial differential equations are very important to understand better the mechanisms of complex physical phenomena. Several methods have been proposed for studying the analytical solutions of fractional partial differential equations. Among these are the variational iteration method [16−18], the Adomian descomposition method [19, 20], the fractional sub-equation method [12−13], the homotopy perturbation technique [14−17]. The searching of new analytical solutions for fractional partial differential equations is an important topic, which can also provide valuable reference for other related research. The homotopy analysis method (HAM), [18−13] transforms a problem into an infinite number of linear problems without using the perturbation techniques, this method employs the concept of the homotopy from topology to generate a convergent series solution. The HAM was applied to solving the fractional heat-like partial differential equations subject to the Neumann boundary conditions [12] and fractional diffusion-wave equations [20]. The authors in [14] solved different linear and nonlinear systems of fractional partial differential equations, using the HAM. The Laplace homotopy perturbation method (LHPM) is a combination of the homotopy analysis method proposed by Liao in 1992 and the Laplace transform [15, 16].

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Various authors have proposed several schemes to solve fractional partial differential equations with Liouville-Caputo and Caputo-Fabrizio fractional operators. Dehghan in [17] applied the HAM to solve linear partial differential equations, in this work, fractional derivatives are described in the Liouville-Caputo sense. Xu in [18] studied analytically the time fractional wave-like differential equation with a variable coefficient, the author reduced the governing equation to two fractional ordinary differential equations. Jafari in [39] used the HAM to obtain the solution of multi-order fractional differential equation studied by Diethelm and Ford [20]. Goufo et al. [19] developed a mathematical analysis of a model of rock fracture in the ecosystem and applied the CF fractional derivative, where analytical and computational approaches are obtained. Other analytical approaches that could be of interest are presented in [18].

Laplace transform which is one of the foundations of applied mathematics enables the transformation of a differential equation into an algebraic equation which makes it easier to solve [2]. In the case of Caputo FDEs, it is especially good, giving the precise solutions to linear systems with constant coefficients [3]. The best solutions are discussed in this paper with references to the latest investigations on the precise solutions [4], [5].

2. Mathematical Preliminaries

2.1 The Laplace transform of the Caputo fractional derivative is given below:

The Laplace transform forms a major part of calculations and solutions to the fractional differential equations (FDEs). It provides an intermediation between formulations that exist in the time domain, where the influence of memory effects and nonlocal operators dominate, and the intricate frequency-domain, where algebraic manipulations become practical.

Given a well-behaved function which is sufficiently well behaved.

f(t), $t \geq 0$ Where: t denotes time and Laplace transform is:

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \Re(s) > 0.$$

This transform is an integral transform that has the effect of mapping convolution in the time domain to multiplication in the frequency domain where it is especially useful when applying operators that have a convolution kernel associated with them like the fractional derivatives and integrals.

2.1.1 Caputo Fractional Derivative and Its Laplace Transform

Let us recall that the Caputo fractional derivative of order α\alphaα (with $m-1<a<m, m \in N$) is defined as:

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

This operator is often preferred in applied problems because it allows the formulation of initial conditions in terms of integer-order derivatives of f(t), i.e., f′(0),f′′(0),f(0),...,$f^{(m-1)}(0)$, which are physically interpretable.

The Laplace transform of the Caputo derivative is given by the formula:

$$L\{^{C}D^{\alpha}f(t)\}=s^{\alpha}F(s)-\sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0),$$

where F(s)=L{f(t)}.

The latter result can be drawn out in two critical observations:

The convolution of fractional differentiation: the kernel.

$(t-\tau)^{-\alpha}$ the occurrence in the integral definition may be treated by Laplace convolution theorems.

Identity of gamma functionality: the scaling factor. Γ(m- α) gives consistency with the differentiation of integer order in case α→m.

In this way the Caputo operator is the natural generalization of the standard derivative and the Laplace domain representation is a compact algebraic relation.

2.1.2 Connection with the Riemann–Liouville Fractional Integral

In fractional calculus, the Riemann–Liouville integral of order $\alpha>0$ is defined as:

$$(J^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Its Laplace transform is particularly simple:

$$L\{J^{\alpha} f(t)\} = s^{-\alpha} F(s).$$

The inverse relation between differentiation and integration (fractional sense) is manifested in this identity. s^{α} the Laplace transform equivalent of is the fractional differentiation, and the other way round. $s^{-\alpha}$ is equivalent to fractional integration.

Therefore, the Laplace transform offers the overarching structure in which the operators of the fractional nature can be reduced to the algebraic powers of s.

2.1.3 Example: First-Order Caputo Derivative

To illustrate, consider $f(t)=e^{at}$, with $a \in R$. Its Laplace transform is:

$$F(s) = \frac{1}{s-a}, \Re(s) > a.$$

The Caputo derivative of order α satisfies:

$$L\{{}^{C}D^{\alpha} e^{at}\} = s^{\alpha} \cdot \frac{1}{s-a} - \sum_{k=0}^{m-1} s^{\alpha-k-1} a^k \, .$$

For instance, with $0<\alpha<1$, this reduces to:

$$L\{{}^{C}D^{\alpha} e^{at}\} = \frac{s^{\alpha}}{s-a} - s^{\alpha-1}.$$

This identity may be inverted to provide a direct MittagLefler function representation in the time domain, which demonstrates the use of Laplace methods to provide ready access to closed-form solutions to fractional equations.

2.1.4 Connection with the Riemann–Liouville Fractional Integral

The Laplace transform formula for the Caputo derivative is essential in solving linear FDEs with constant coefficients. Consider the canonical problem:

$$^{C}D^{\alpha}y(t)+\lambda y(t)=g(t),\ y^{(k)}(0)=yk,\ k=0,1,\ldots,m-1.$$

Applying the Laplace transform yields:

$$s^{\alpha}Y(s)-\sum_{k=0}^{m-1} s^{\alpha-k-1}\, yk+\lambda Y(s)=G(s).$$

Rearranging:

$$Y(s)=\frac{G(s)}{s^{\alpha}+\lambda}+\frac{\sum_{k=0}^{m-1} s^{\alpha-k-1}\, yk.}{s^{\alpha}+\lambda}$$

This expression is a solved form of algebraic solution in the Laplace domain, which can be inverted, usually using Mittag-Leffler functions. It would be far more complicated to derive these explicit forms without the Laplace framework.

2.1.5 Variants: The Laplace–Carson Transform

Recent investigations (e.g., Kumar and Qureshi [4]) point to the usefulness of the LaplaceCarson transform, which is defined by:

$$LC\{f(t)\}=\int_0^{\infty} e^{-st}f(t)dt-\frac{f(0)}{s}.$$

This version alters the kernel to ease the work with some types of FDEs, especially when solutions to FDEs deal with singular kernels or generalized initial conditions. Its ability to reduce to the conventional Laplace method with an error, strengthens the power of the transform-based ways of the fractional calculus.

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2.1.6 Broader Implications

The use of the Laplace domain formulation of the Caputo derivative does more than just theoretical analysis:

Engineering systems: Laplace transforms are commonly used to solve viscoelastic models, and fractional-order control systems as well as anomalous diffusion equations.

Mathematical physics Fractional diffusion-wave equations and fractional Schrodinger equations are based on the use of fractions. $s\alpha$-scaling during spectral analysis.

Numerical validation Laplace-based solutions can be used to test the numeric schemes, e.g. Grunwald-Letnikov approximations or predictor corrector schemes.

The Laplace transform, therefore, does not only make algebraic manipulations easier, but also gives physical interpretation, relating fractional operators to the effects of memory and hereditary aspects in real-world systems.

2.2 Mittag–Leffler Function

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2.2.1 Definition and Properties

The two-parameter Mittag–Leffler function is defined as

$$E\alpha,\beta(z) = \sum_{n=0}^{\infty} \frac{zn}{\Gamma(\alpha n + \beta)}, \alpha > 0, \beta > 0,$$

where Γ(·) denotes the Gamma function.

When α=1, β=1, this reduces to the exponential:

$$E_{1,1}(z)=e^{z}.$$

When α=2, β=1, it is similar to the hyperbolic cosine function.

More generally, varying α and The β gives MittagLeffler the ability to smoothly transition between exponential-like growth and power-law decay, expressing a range of behaviors between weakly-local dynamics and highly-memory-dependent dynamics.

The ability to do so is why the role is so necessary in the modeling of fractions.

2.2.2 Laplace Transform Identity

The rank of the Mittag–Leffler function in solving Caputo FDEs becomes evident through its Laplace transform:

$$L\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha-\lambda}}, |\ \lambda\ |<|\ s\ |^{\alpha}.$$

This identity demonstrates the natural occurrence of the function when the expressions, which are inverted through Laplace-domain, contain s with various powers which are fractions. In the standard equation of fractional relaxation given as an example:

$$^{C}D^{\alpha}y(t)+\lambda y(t)=0, y(0)=1,$$

the Laplace-domain solution is Y(s) = $\frac{s^{\alpha-1}}{s^{\alpha+\lambda}}$. Inverting this via the above formula yields:

$$y(t)=E\alpha(-\lambda t^{\alpha}),$$

representative that the Mittag–Leffler purpose is the fractional analog of the exponential decay law.

2.2.3 Applications Interpretation in Applications.

The MittagLeffler function is not only a mathematical oddity but it conveys a strong physical significance.

Fractional relaxation and diffusion: Fractional dynamics replace classical exponential decay Markovian dynamics and instead the system is characterized by a slower and stretched decay, which is described by $E_{\alpha}(-t^{\alpha})$. This shows effects of memory when the system would remember its past with more strength than in exponential case.

Viscoelastic materials: In some cases, the stress strain relation of polymers and biological tissues is power law-type relaxation and this is naturally described by Mittag-Leffler functions.

Oscillatory systems: Duan et al. [6] illustrate the utilisation of Mittag-Leffler functions in equations of fractional oscillation, in which they specify oscillations with damping patterns not representable by sine and cosine functions.

Control theory and engineering Fractional-order controllers, including the $PI^{\lambda}D^{\mu}$ controller, frequently are based on solutions written in the form of Mittag-Leffler functions that are useful in tuning systems whose behavior is memory-dependent and hereditary.

The role of these areas fills the discontinuity between exponential behaviour (memoryless, the purely local dynamics) and power-law behaviour (strongly nonlocal, the history-dependent dynamics).

2.2.4 Asymptotic Behavior

The other detail is also the asymptotic property of the MittagLeffler function. It is exponentially linear in small arguments $|z|$, but exponential in large arguments:

- Near zero: $E_{\alpha,\beta}(z) \approx \frac{1}{\Gamma(\beta)} + \frac{z}{\Gamma(\alpha+\beta)} + \cdots.$
- For large |z|: $E_{\alpha,\beta}(z) \sim -\sum_{k=1}^{N} \frac{z^{-k}}{\Gamma(\beta-\alpha k)}.$

This two-sidedness, exponential at the origin and power-law at infinity, is why it is such an effective model of the real-world processes which start with the high dynamics, but then have long-tail memory effects.

2.2.5 Broader Significance

The increasing role of the MittagLeffler in applied mathematics is an indication of the increased interest in the field of fractional-order models in science and engineering. Nowadays it is a common instrument in the fields of:

- In biophysics, Fick law models are not sufficient to describe anomalous diffusion in cells.
- Finance, in which memory effects in market volatility are calculable via the use of fractional stochastic differential equations.
- Signal processing Signal processing is a domain where Mittag Leffler kernels are found to be a natural description of long memory noises.

Therefore, the exponential function supports the theory of classical differential equations in the same way that the Mittag Leffler function supports the theory of the fractional differential equations.

3. Method for Solving Caputo Fractional Differential Equations

The Caputo fractional derivative presents a mathematically sound but physically significant means of extrapolating classical models to the fractional realms. The Laplace transform method gives the analytical solution, unlike the numerical schemes which depend on discretization, and it therefore makes it especially appealing to theoretical studies and benchmark comparison.

We take into consideration the general linear Caputo fractional differential equation (FDE):

$$ {}^{C}D^{\alpha}y(t)+a\,{}^{C}D^{\beta}y(t)+by(t)=g(t), t\geq 0, $$

where $0<\beta<\alpha\leq m$, $m\in N$, and $g(t)$ is a given forcing function. Initial conditions are prescribed in terms of integer-order derivatives:

$$y^{(k)}(0)=ck, k=0,1,\dots,m-1.$$

The formulation is general enough to encompass the oscillators of fractional nature, relaxation and dynamics of viscoelastic nature. The process of solution has four primary steps, which are systematic.

3.1 Step 1: Applying the Laplace Transform

Using the result from Section 2.1, the Laplace alter of the Caputo derivative is expressed as:

$$L\{^{C}D\alpha y(t)\}=s^{\alpha}Y(s)-\sum_{k=0}^{m-1} s^{\alpha-k-1}\, y^{(k)}(0).$$

Applying this to each period in the equation stretches:

$$s^{\alpha}Y(s)-\sum_{k=0}^{m-1} s^{\alpha-k-1}\, ck + a\left[s^{\beta}Y(s)-\sum_{k=0}^{n-1} s^{\beta-k-1}\, ck\right]+bY(s)=G(s),$$

$$\text{where } Y(s)=L\{y(t)\}.$$

This transformation changes the integro-differential problem to an algebraic one in the Laplace domain making the analysis largely easier.

3.2 Step 2: Solving for Y(s)

Reorganizing terms, we obtain:

$$Y(s)=\frac{G(s)+\sum_{k=0}^{m-1} s^{\alpha-k-1}\, ck + a\sum_{k=0}^{n-1} s^{\beta-k-1} ck}{s^{\alpha}+as^{\beta}+b}.$$

There are two contributions made with reference to this formula:

Forcing response - this is dictated by G(s), and it is the way the external input drives the system.

Initial-condition response — is a term representing the sums of ck which represent the inherent memory and stored energy of the system.

When g(t) is homogeneous, i.e. g(t) =0, the numerator becomes a combination of terms that are entirely dependent on initial conditions. This situation is especially significant when one wants to study natural vibrations, damping, and stability characteristics.

3.3 Step 3: Inverse Laplace Transform

The critical step lies in upsetting the Laplace transform to recuperate y(t). The denominator $s^{\alpha}+as^{\beta}+b$ removes the need to invert by table, but the form is vulnerable to Mittag Leffler pictures.

For example, terms of the form:

$$\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}$$

can be upturned through series growth. One general form is:

$$y(t)=t^{\alpha-\gamma-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-b)^{j}\,(-a)^{k}\binom{j+k}{k}\frac{t^{k(\alpha-\beta)+j\alpha}}{\Gamma(k(\alpha-\beta)+(j+1)\alpha-\gamma)}.$$

Although such expansion offers an overall solution, even simpler cases are reduced to closed-form Mittag-Leffler functions more convenient analytically and numerically.

In the case of single-term FDEs such ${}^{C}D^{\alpha}y+\lambda y=0$, the solution is simply:

$$y(t)=c_0E_{\alpha}(-\lambda t^{\alpha}).$$

In the case of two-term systems, one has a blend of generalized MittagLeffler functions, an extension of the exponentialoscillatory behavior of classical ODEs.

3.4 Step 4: Interpretation and Applications.

The LaplaceMittagLeffler model demonstrates some significant characteristics of Caputo FDE solutions:

Effects of memory: As opposed to exponential solution, which decays or grows exponentially, the MittagLeffler solutions exhibit stretched decay or long tailed oscillations, as it is in memory-dependent systems.

Sensitivity of parameters: The exponents α and β inhibit the pace of memory loss. Lower order relates to a robust memory, whereas an increase in order relates to classical exponential responses.

Practical relevance:

- Ali et al. [5] give direct formulas of a two-term fractional equation, and it is found that Mittag-Leffler functions are generalisation of the sine, cosine, and exponential functions.
- Shen et al. [10] use the framework on fractional oscillators, which describe damping profiles that are not described by classical models.
- Atanackovic et al. [9] point out that it is used in viscoelasticity, where the fractional models are appropriate to describe actual material behavior.

Therefore, the approach is not only capable of generating precise solutions but it also provides physical interpretability.

3.5 Advantages of the Method

Laplace transform method of solving Caputo FDEs has certain advantages over other methods:

- Precision - solutions are gotten with no truncation and discrete errors.
- Universality The same structure is true to both homogeneous and non-homogeneous equations, arbitrary forcing functions.
- Analytic continuation Laplace-domain solutions can be analysed easier in the long term (asymptotically), providing information about the long-term behaviour.
- Relation to classical theory When solutions of equations of the form when .

This technique is therefore the analytic method of choice in fractional modeling as it offers sufficient theoretical rigor and practical reliability as observed by a number of researchers [4], [11], [20].

4. Advantages of the Method

Example 1: Similar Fractional Equation

Solve ${}^{C}D^{3/2}y(t)+y(t)=0$, $y(0)=1$, $y'(0)=0$.

Laplace: $s^{3/2}Y(s)-s^{1/2}+Y(s)=0$.

Solve: $Y(s) = \frac{s^{1/2}}{s^{3/2}+1}$.

Inverse: $y(t)=E_{3/2,1}(-t^{3/2})$, reliable with Kazem's answers [1] and oscillator subtleties [6], [8].

Example 2: Bagley-Torvik Equation

Solve ${}^{C}D^{2}y(t)+{}^{C}D^{3/2}y(t)+y(t)=1+t$, $y(0)=1$, $y'(0)=1$.

Laplace: $(s^2+s^{3/2}+1)Y(s)=\frac{1}{s}+\frac{1}{s^2}+s+1$.

Solve: $Y(s)=\frac{1}{s}+\frac{1}{s^2}$.

Inverse: y(t)=1+t, aligning with Ali et al. [5] and mechanical applications [2], [13].

Example 3: Slight Oscillator Equation

Solve ${}^{C}D^{\alpha}y(t)+a{}^{C}D^{\alpha/2}y(t)+by(t)=8 \quad 0<\alpha\leq 1$, $y(0)=0$.

Laplace: $(s^{\alpha}+as^{\alpha/2}+b)Y(s)=\frac{8}{s}$.

Inverse: $y(t) = 8\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(at^{\alpha/2})n(bt^{\alpha})^k}{\Gamma(\alpha n+\alpha k+1)}$, as derived by Duan et al. [11] and supported by oscillator studies [7], [9], [10].

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5. Analysis and Advantages

The Laplace transform technique has come out as one of the most potent methods in the solving of Caputo-type fractional differential equations (FDEs). It is attractive in that it transforms problems based on fractional derivatives - operators that are necessarily defined using nonlocal integrals - into complex frequency domain algebraic equations. This simplifies the mathematical treatment as well as permits the exact solutions to be expressed as special functions well understood, like the MittagLeffler function.

5.1 Precision and Analytical purity.

The fact that the Laplace transform method can give closed-form solutions is one of its major strengths. In the case of linear equations, it is quite common to express solutions in a direct form using MittagLeffler functions or similar expansion of series. Here is in contrast to purely numerical derivations, like Grunwald-Letnikov approximations or finite-difference discretisations, which are based on a time-axis discretisation, and may cause truncation errors to build up.

As an example, in the fractional relaxation equation, the Laplace transform would be used directly to obtain:

$$y(t)=E_{\alpha,1}(-\lambda t^{\alpha}),$$

a small solution which concurrently captures short-time exponential-like behavior, coupled with long-time power-law decay. The existence of these forms of the expression allows obtaining asymptotic properties. It is indeed known that at large t,

$$E_{\alpha,1}(-\lambda t^{\alpha}) \sim \frac{1}{\Gamma(1-\alpha)\lambda t^{\alpha}},$$

which gives direct information on the long-memory effects of fractional dynamics [4], [6]. It would be much harder to perform this kind of asymptotic analysis by numerically means alone.

5.2 Versatility Across Models

The application of the Laplace method is far more helpful than simple relaxation problems. It is general to a degree that it can be used on higher-order, multi-term and even integro-differential systems.

fractional telegraph equations: The Laplace framework was applied to fractional extensions of the telegraph equation by Khan et al. [3] who demonstrated that, in this case, the exact propagator functions can be obtained, phenomena of anomalous propagation of waves.

Fractional oscillators: The method was applied to the fractional oscillator models by Lim and Teo [8], Shen et al. [10] and others. These equations model systems in which the restoring force and damping take into account memory effects and the LaplaceMittagLeffler solutions reveal complicated oscillatory behavior in addition to classical sinusoidal reactions.

Fractional logistic equations Fractional logistic equations have been extended to population models, including the fractional logistic equation [12], which points out how the technique can support nonlinear growth dynamics through first linearizing or breaking them down into Laplace-solvable form.

Integro-differential systems: Laplace transform can also be applied to systems which form a mixture of differentiation and convolution integrals, and it is especially useful with hereditary systems in viscoelasticity and heat conduction [16].

This generality has shown that the approach is not restricted to a small group of FDEs but rather provides a general framework of a fractional model.

5.3 Comparison to Numerical Methods.

Although numerical methods are still necessary in the case of nonlinear or variable-coefficient problems, another method, the Laplace transform method has obvious merits in some situations:

- Elimination of discretization errors: There are no finite sum approximations of derivatives to be made, and so the solutions are represented by precise special functions, which is why discretization errors are avoided.

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- Analytic expansions: The closed expressions are highly accurate at long time or short time scales, which cannot be easily extracted using raw numerical information.
- Benchmarking Benchmark solutions computed by Laplace techniques give a reference point against which numerical codes can be verified.

Complexity of inverse transforms is however one of the challenges. Inversion with denominators of more than one power of a fraction can result in nested series expansions, or a generalised version of the MittagLeffler functions, which is not always easy to compute. However, in linear systems, such inversions are still manageable and their computation has been facilitated by symbolic computing packages.

5.4 Theoretical and Practical Impact.

Recent studies also highlight the theoretical richness and practical extensiveness of the Laplace transform methodology:

- Sequential FDEs: Vatsala et al. [13] demonstrate that the technique is applicable to equations with sequential Caputo derivatives, where orders in cascaded combinations are used.
- Integro-differential systems Its capacity to solve systems involving integral constraints together with a combination of fractional derivatives is extended by Kaplan [15] and others.
- In physics and engineering: Laplace-based solutions With applications: Since anomalous diffusion in porous media is a model of viscoelastic beam equations, solutions can be expressed explicitly as functions of parameters of the system.

Mathematical validation: Some of the studies like the one by Atanackovic et al. [9] and the follow up studies [11], [17] through to [20] underline the [LaplaceMittagLeflower] solutions as being consistent with both experimental and theoretical data.

5.5 Summary of Advantages

In order to sum up the Laplace transform method has the following advantages:

- Precision: closed-form or series solutions without error of approximation.
- Versatility: relaxation, oscillators, telegraph models, logistic growth and integro-differential systems.
- Insight: asymptotic behavior is analyzable and also memory effects are analyzable.
- Basic value: is used as a standard of number schemes and is used in further theoretical elaboration.

Although the inversion of the complex Laplace-domain expressions can be still challenging, the benefits are more than the complications. In the case of the linear Caputo FDEs, the method is perhaps the most robust and explanatory tool of analysis that is currently in existence.

6. Conclusion

The Laplace transform approach has a powerful platform of exact solutions of Caputo FDEs, with an extensive literature support base [1]20]. The fact that it can deal with initial conditions and provide analytical solutions makes it the best choice of theoretical and applied research especially in the area of fractional oscillator systems and associated problems [6]12. The work could be continued in the future with nonlinear FDEs or hybrid transforms such as Laplace-Carson [4] current studies are based on.

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